October Math Gems

PROBLEM OF THE WEEK 18

§1 problems

Problem 1.1. Prove the proposition: If a side of a triangle is less than the average (arithmetic mean) of the two other sides, the opposite angle is less than the average of the two other angles.

answer. Since $A+B+C=180^\circ$, proving that $A<(B+C)/2$ amounts to proving that $A<\left(180^\circ-A\right)/2$ or $A<60^\circ$. But $A<60^\circ$ is equivalent to $\cos A>1/2$, which suggests the use of the law of cosines.

By hypothesis, $b+c>2a$. Squaring both sides and applying the law of cosines, we obtain

$$b^2+2bc+c^2>4\left(b^2+c^2-2bc\cos A\right)$$

or

$$8bc\cos A>3b^2+3c^2-2bc.$$

Subtracting $4bc$ from both sides, we obtain

$$4bc(2\cos A-1)>3(b-c)^2\geqq 0.$$

Therefore $\cos A>1/2$. □

Problem 1.2. Prove that the only solution of the equation

$$x^2+y^2+z^2=2xyz$$

in integers x,y, and z is $x=y=z=0$.

answer. Suppose x,y, and z are integers. Let 2^k, with $k\geqq 0$, be the highest power of 2 that divides x,y, and z, so that $x=2^kx',y=2^ky'$, and $z=2^kz'$. Then substituting in the given equation and dividing through by 2^{2k}, we obtain

$$\left(x'\right)^2+\left(y'\right)^2+\left(z'\right)^2=2^{k+1}x'y'z'.$$

Since the right-hand side is even, so is the left-hand side, and either x',y', and z' are all even or just one of them is. But if x',y', and z' are not all zero (and if one is, the others are), they cannot all be even, because 2 is not a common factor. Suppose x' is even and y' and z' are odd. Subtracting $(x')^2$ from both sides of the above equation yields

$$\left(y'\right)^2+\left(z'\right)^2=x'\left(2^{k+1}y'z'-x'\right).$$

Both $(y')^2$ and $(z')^2$ are of the form $4n^2+4n+1$, and so the left-hand side divided by 4 leaves the remainder 2 , whereas the right-hand side is divisible by 4 (both x' and the quantity in parenthesis are even): A Contradiction. □

Problem 1.3. Show that each number of the sequence

$$49, 4489, 444889, 44448889, \ldots$$

is a perfect square.(Recall the formula for the sum of a geometric progression)

answer. Recall the formula for the sum of a geometric progression. The number of the sequence that has $2n$ digits is

$$\begin{aligned} 9+8(10 & +10^2+\cdots+10^{n-1}) \\ & +4\left(10^n+10^{n+1}+\cdots+10^{2n-1}\right) \\ & =1+\left(8+4 \cdot 10^n\right)\left(1+10+\cdots+10^{n-1}\right) \\ & =1+4\left(10^n+2\right)\left(10^n-1\right)/(10-1) \\ & =\left(\frac{2 \cdot 10^n+1}{3}\right)^2 \end{aligned}$$

This is the square of an integer, since

$$\left(2 \cdot 10^n+1\right)/3=1+6\left(10^n-1\right)/9=666 \cdots 67,$$

a number with n digits. □

Problem 1.4. Solve the system

$$\begin{aligned} & 2x^2-4xy+3y^2=36 \\ & 3x^2-4xy+2y^2=36 \end{aligned}$$

(One solution is easy to guess, but you are required to find all solutions. Knowledge of analytic geometry is not needed to solve this problem, but may help to understand the result-how?)

answer. We are required to find the points of intersection of two congruent ellipses symmetrical to each other with respect to the line $x=y$. Subtraction of the equations yields $x^2=y^2$. There are four points of intersection: $(6,6),(-6,-6),(2,-2),(-2,2)$. □

Problem 1.5. Solve the following system of three equations for the unknowns x, y, and z (a, b, and c are given) :

$$\begin{aligned} x^2y^2+x^2z^2 & =axyz, \\ y^2z^2+y^2x^2 & =bxyz, \\ z^2x^2+z^2y^2 & =cxyz. \end{aligned}$$

answer. If $x=0$, the second (or third) equation yields $y^2z^2=0$, and so one more unknown, y or z, must also be 0 . Hence either x, y, and z are all different from 0 or at least two vanish. If any two vanish, the equations are satisfied.

Now we consider the case in which no one of the three unknowns is 0 . By dividing, we obtain the system

$$\begin{aligned} \frac{zx}{y}+\frac{xy}{z} & =a \\ \frac{yz}{x}+\frac{xy}{z} & =b, \\ \frac{yz}{x}+\frac{zx}{y} & =c \end{aligned}$$

Adding these three equations and dividing by 2, we have

$$\frac{yz}{x}+\frac{zx}{y}+\frac{xy}{z}=\frac{a+b+c}{2}.$$

From this equation we subtract each of the three equations of the foregoing system and obtain

$$\begin{aligned}\frac{yz}{x}&=\frac{-a+b+c}{2}\\ \frac{zx}{y}&=\frac{a-b+c}{2}\\ \frac{xy}{z}&=\frac{a+b-c}{2}.\end{aligned}$$

The product of these three equations is

$$xyz=(-a+b+c)(a-b+c)(a+b-c)/8$$

which we divide by each equation of the foregoing system to obtain, after extracting a square root,

$$\begin{aligned}x&=[(a-b+c)(a+b-c)]^{1/2}/2\\ y&=[(-a+b+c)(a+b-c)]^{1/2}/2\\ z&=[(-a+b+c)(a-b+c)]^{1/2}/2.\end{aligned}$$

We must take into account, however, the two values of each square root. Let us concentrate upon a suggestive particular case and assume that a, b, and c are the lengths of the three sides of a triangle. Then by $(^\circ)$ above, xyz is positive, and therefore only the following four combinations of signs are admissible:

$$\begin{array}{ll} x & ++-- \\ y & +-+- \\ z & +--+ \end{array}$$

□

Problem 1.6. Prove: If n is an integer greater than $1, n^{n-1}-1$ is divisible by $(n-1)^2$.

answer. We observe first that if n is greater than 1 , the quotient of $n^{n-1}-1$ and $n-1$ is

$$n^{n-2}+n^{n-3}+\cdots+n+1.$$

We conceive this sum as resulting from the polynomial

$$R(x)=(1+x)^{n-2}+(1+x)^{n-3}+\cdots+(1+x)+1$$

when we substitute in it $n-1$ for x. In the expansion of $R(x)$ in powers of x (you may use the binomial formula), the term independent of x is $R(0)=n-1$, and so

$$R(x)=Q_{n-3}(x)x+n-1$$

where $Q_{n-3}(x)$ is a polynomial of degree $n-3$ whose coefficients are integers. Now we substitute $n-1$ for x and collect our conclusions:

$$\begin{aligned}n^{n-1}-1&=(n-1)R(n-1)\\ &=(n-1)\left[Q_{n-3}(n-1)(n-1)+n-1\right]\\ &=(n-1)^2\left[Q_{n-3}(n-1)+1\right].\end{aligned}$$

□

Problem 1.7. Ten people are sitting around a round table. The sum of ten dollars is to be distributed among them according to the rule that each person receives one half of the sum that his two neighbors receive jointly. Is there just one way to distribute the money? Prove your answer.

answer. Let $A, B, C, D, \ldots, J$ be the persons around the table, and $a, b, c, d, \ldots, j$ the amounts received by them, respectively; B is to the right of A, C to the right of $B, \ldots, A$ to the right of J. The rule is expressed by the equations $b = \frac{a+c}{2}, \quad c = \frac{b+d}{2}, \quad d = \frac{c+e}{2}, \ldots, \quad a = \frac{j+b}{2}$. First solution. From the above equations, it follows that

$$b - a = c - b = d - c = \ldots = a - j$$

so that everyone's share exceeds that of his neighbor on the left by the same amount. This constant excess must be zero, since

$$(b - a) + (c - b) + (d - c) + \ldots + (a - j) = 0.$$

There is just one way to distribute the money: all shares are equal. Second solution. Some person (or persons) must receive the maximum amount. Let such a person be B. Then none of the numbers $a, \ldots, j$ is greater than b; and, in particular,

$$b - a \geqq 0, \quad b - c \geqq 0.$$

Yet, by the condition,

$$b - a = -(b - c).$$

Consequently, both of the two numbers $b - a$ and $b - c$ must be zero. Thus c also attains the maximum, as does d, and so on. Therefore $a = b = c = \ldots = j$. □

Problem 1.8. If the coefficients of x^{-2} and x^{-4} in the expansion of

$$\left(x^{\frac{1}{3}} + \frac{1}{2x^{\frac{1}{3}}}\right) \quad (x > 0)$$

are m and n respectively, then compute $\frac{m}{n}$.

answer. First, we need to find the general term:

$$\begin{aligned} T_{r+1} &= [18]r \left(x^{\frac{1}{3}}\right)^{18-r} \left(\frac{1}{2x^{\frac{1}{3}}}\right)^r \\ &= [18]r x^{6-\frac{2r}{3}} \frac{1}{2^r} \end{aligned}$$

So, to get the coefficients of x^{-2} we need to assume:

$$6 - \frac{2r}{3} = -2$$

$$r = 12$$

Doing the same for x^{-4}

$$6 - \frac{2r}{3} = -4$$

$$r = 15$$

$\frac{\text{coefficient of } x^{-2}}{\text{coefficient of } x^{-4}} = \frac{[18]12\frac{1}{2^{12}}}{[18]15\frac{1}{2^{15}}} = 182$ □

Problem 1.9. The coefficients of x^{50} in the expansion of

$$(1+x)^{1000} + 2x(1+x)^{999} + 3x^2(1+x)^{998} + \cdots + 1001x^{1000}$$

is [1002]50. Prove or Disprove.

answer. Let:

$$S = (1+x)^{1000} + 2x(1+x)^{999} + 3x^2(1+x)^{998} + \cdots + 1001x^{1000} \quad \textit{[1]}$$

$$\frac{x}{1+x}S = x(1+x)^{999} + 2x^2(1+x)^{998} + \cdots + 1000x^{1000} + \frac{1001x^{1001}}{1+x} \quad \textit{[2]}$$

Now, subtract *[1]* and *[2]* to get the following:

$$\left(1 - \frac{x}{1+x}\right)S = (1+x)^{1001} + x(1+x)^{999} + \cdots + x^{1000} - \frac{1001x^{1001}}{1+x}$$

$$S = (1+x)^{1001} + x(1+x)^{1000} + x^2(1+x)^{999} + \cdots + x^{1000} - 1001x1001$$

Notice that this is a sum of geometric pattern.

$$S = (1+x)^{1002} - x^{1002} - 1002x^{1001})$$

So the coefficient of x^{50} is [1002]50 □

Problem 1.10. Suppose x and y are nonzero real numbers simultaneously satisfying the following system of equations

$$x + \frac{2018}{y} = 1000$$

$$\frac{9}{x} + y = 1$$

Find the maximum possible value of $x + 1000y$.

answer. First, we need to multiply the first equation with y, and the second with x to obtain the following:

$$xy + 2018 = 1000y$$

$$9 + xy = x$$

Subtracting the two equations yielding:

$$2009 = 1000y - x$$

Now, we need to solve the above equation for y, then substitute it into $9 + xy = x$ yields:

$$x^2 + 1009x + 9000 = 0$$

which factors as:

$$(x+9)(x+1000) = 0$$

This gives us two possible solutions

$$(x, y) = (9, 2)$$

$$(x, y) = (-1000, \frac{1009}{1000})$$

Then the requested sum is $-9 + 1000 \cdot 2 = 1991$ □

Problem 1.11. From the following system of equations

$$x^2 - y^2 = 9$$

$$xy = 3$$

The value of of $x + y$ can be written in the form of $\pm\sqrt{\sqrt{a} + b}$, then find the values of a and b.

answer. First, we need to solve for y in the second equation and substitute in the first one yielding:

$$x^2 - \left(\frac{3}{x}\right)^2 = 9$$

$$x^4 - 9x^2 - 9 = 0$$

$$x^2 = \frac{9 + \sqrt{117}}{2}$$

$$x = \pm\sqrt{\frac{9 + \sqrt{117}}{2}}$$

Thus, we can use this in finding the value of y.

$$y = \pm\sqrt{\frac{9 - \sqrt{117}}{2}}$$

So we get that,

$$x + y = \pm\left(\sqrt{\frac{9 + \sqrt{117}}{2}} + \sqrt{\frac{9 - \sqrt{117}}{2}}\right)$$

Unfortunately, we're not done yet, we still need to acquire the form of $\sqrt{\sqrt{a} + b}$ which requires us to square both sides, yielding:

$$(x + y)^2 = \sqrt{117} + 6$$

$$x + y = \pm\sqrt{\sqrt{117} + 6}$$

Thus, $a = 117$, $b = 6$. □

Problem 1.12. Let a and b be distinct real numbers. Solve the following equation

$$\sqrt{x - b^2} - \sqrt{x - a^2} = a - b$$

answer. It should be obvious that the following conditions must hold true:

$$x \geq a^2 \quad x \geq b^2$$

Actually the simplest approach to solve this equation is taking the conjugate, another approaches leads to rather complicated computations. Taking the conjugate gives

$$\frac{a^2 - b^2}{\sqrt{x - b^2} + \sqrt{x - a^2}} = a - b$$

which is equivalent to

$$\sqrt{x-b^2}+\sqrt{x-a^2}=a+b$$

Adding this to the original equation gives the following:

$$\sqrt{x-b^2}=a$$

This implies that

$$x=\sqrt{a^2+b^2}$$

□

Problem 1.13.

$$\log_2(11-6x)=2\log_2(x-1)+3$$

find the value x.

answer.

$$log_2(11-6x)=2.log_2(x-1)+3$$

$$log_2(11-6x)-(log_2(x-1))^2=3$$

$$log_2\frac{11-6x}{(x-1)^2}=3$$

$$\frac{11-6x}{(x-1)^2}=2^3$$

$$11-6x=8(x-1)^2$$

$$11-6x=8(x^2-2x+1)$$

$$11-6x=8x^2-16x+8$$

$$\therefore 8x^2-10x-3=0$$

$$\therefore x=\frac{-1}{4}\text{ or }\frac{3}{2}$$

If you check the answer you will find that only $\frac{3}{2}$ is the value of x, as $x-1>0$. □

Problem 1.14. Solve

$$cos^2(\frac{x}{2})=cos^2(x)$$

, on the interval $0\le x<2\pi$.

answer.

$$\because cos^2(\frac{x}{2})=cos^2(x)$$

$$\therefore (cos(\frac{x}{2}))^2=cos^2(x)$$

$$\therefore (\pm\sqrt{\frac{1+cos(x)}{2}})^2=cos^2(x)$$

$$\therefore \frac{1+cos(x)}{2}=cos^2(x)$$

$$\therefore 1+cos(x)=2.cos^2(x)$$

$$\therefore 2.cos^2(x)-cos(x)-1=0$$

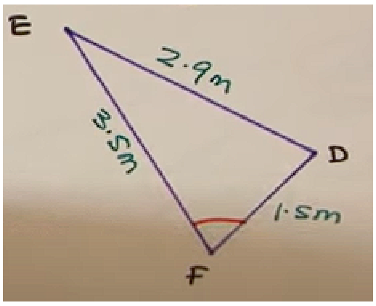
$$\therefore (2cos(x)+1)(cos(x)-1)$$

$\therefore cos(x) = \frac{-1}{2}$ or $cos(x) = 1$

$$\therefore x = 0 \ , \ \frac{4\pi}{3} \text{ and, } \frac{2\pi}{3}$$

□

Problem 1.15. Find angle F



answer. Using the cos law

$$\therefore cos(f) = \frac{d^2 + e^2 - f^2}{2(1.5)(3.5)}$$

$$\therefore cos(f) = \frac{3.5^2 + 1.5^2 - 2.9^2}{2(1.5)(3.5)}$$

$$\therefore cos(f) = 0.58$$

$$\therefore F \simeq 54.5 \text{ degrees}$$

□

Problem 1.16. Determine the range of the function

$$f(x) = 4 - 2sin(3x)$$

answer.

$$\because f(x) = 4 - 2sin(3x)$$

$$\therefore f(x) = -2sin(3x) + 4$$

$\therefore$ the function has an amplitude of 2
But, the +4 at the end of the function means that it will be shifted up by 4.

$$\therefore \text{ the range is } [2, 6]$$

□

Problem 1.17. The surface area of a human's lungs is equal to half of a tennis court (2100 ft^2). How many square feet would the lungs of a 30-person baseball team cover?

answer.

$$\frac{1050ft^2}{1lungs} = \frac{A}{30lungs}$$

$$\therefore A = 31500$$

□

Problem 1.18. At a shop in Times Square one "I LOVE NY" t-shirt is sold every 10 minutes for 19.95 dollars each. The shop opens from 9 am until 9 pm every day.
How many t-shirts are sold in a week?

answer. The shop opens from 9 am to 9 pm, so it opens 12 hours a day.

$$\because 12h = 5040min$$

$\therefore$ the shop opens 5040 min. each week.
$\because$ there is a T-shirt sold every 10 min.
$\therefore$ by making the ratio

$$\frac{1}{10min} = \frac{x}{5040}$$

, then the shop sold 504 shirts every week. □

Problem 1.19.

$$(x+3)(x-5) = 5$$

Find all the solutions to the equation above.

answer.

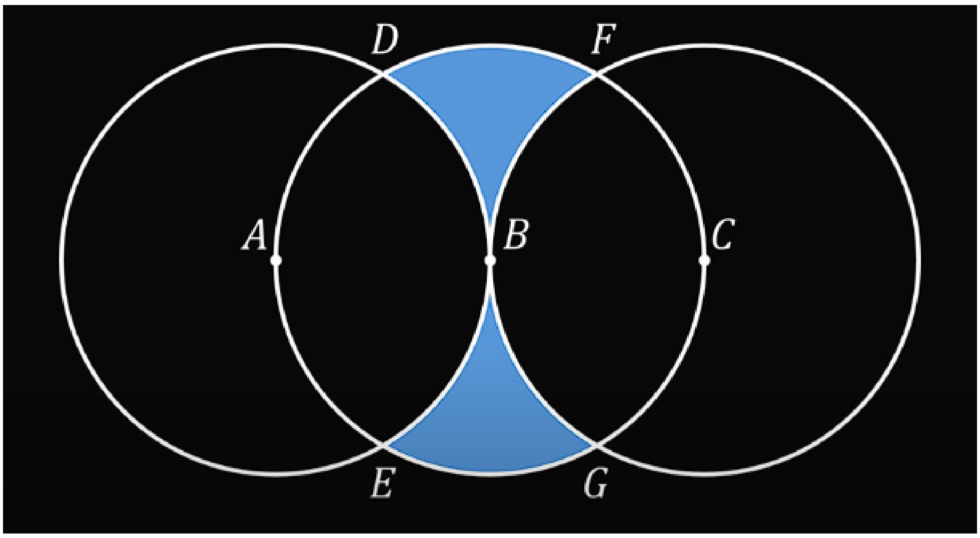
$$\because (x+3)(x-5) = 5$$
$$\therefore x^2 - 2x - 15 = 5$$
$$\therefore x^2 - 2x - 20 = 0$$

$\therefore$ by applying the rule $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

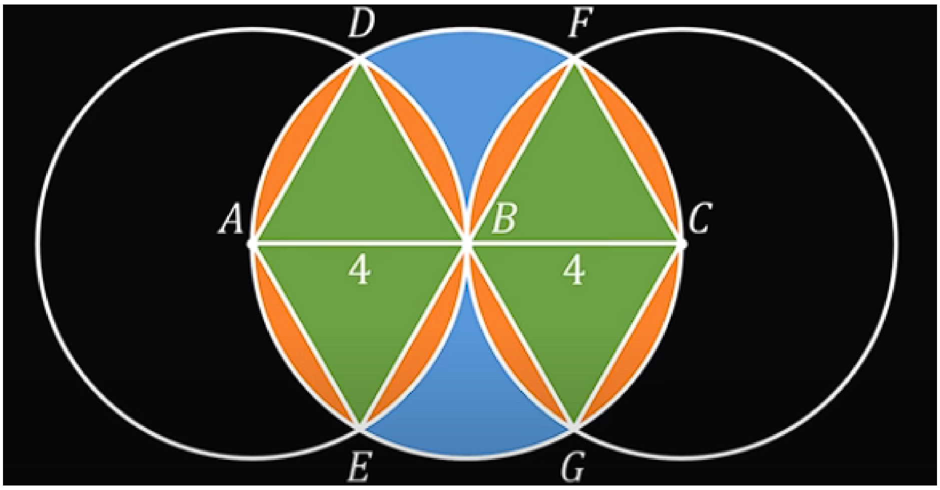
$$\therefore x = 1 \pm \sqrt{21}$$

□

Problem 1.20. Find the area of the blue part, where $r = 4$.

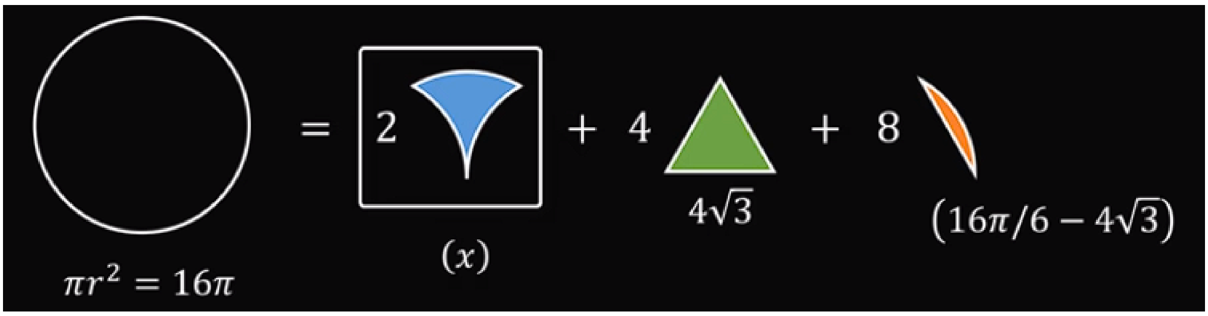


answer. Firstly, connect each circle's center to other centers, intersection points, points of tangency, etc.
Secondly, break down areas into circular sectors and polygons.
After doing the previous, you must end up with this shape:



The area of the orange part is equal to the difference between the area of the sector and the area of the green triangle, as the following figure shows:

The area of the white sector is equal to: $\pi(4^2).\frac{60}{360} = \frac{16\pi}{6}$
The area of the green triangle is equal to $\frac{4^2\sqrt{3}}{4} = 4\sqrt{3}$.
$\therefore$ the area of the orange part is equal to $\frac{16\pi}{6} - 4\sqrt{3}$
Now, to calculate the area of the blue part we will use the fact that the middle circle area is equal to the sum of the areas of the blue, yellow, and green parts.
As we know the area of the circle is equal to πr^2
So, we will use the following equation in order to calculate the area of the blue part:



$\therefore 16\pi = x + 4(4\sqrt{3}) + 8(\frac{16\pi}{6} - 4\sqrt{3})$
$16\pi = x + 16\sqrt{3} + \frac{128\pi}{6} - 32\sqrt{3}$
$0 = x - 16\sqrt{3} + \frac{32\pi}{6}$

$$\therefore x = 16\sqrt{3} - \frac{16\pi}{3} \simeq 10.958$$

□